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Baroclinic instability in a reduced gravity, three-dimensional, quasi-geostrophic model

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The classical quasi-geostrophic model in an active layer with an arbitrary vertical structure is modified by adding a boundary condition at the interface with a passive (motionless) lower layer: the difference between isopycnal and interface elevations is a Lagrangian constant, so that a particle in this boundary remains there and conserves its density. The new model has the appropriate integrals of motion: in particular, a free energy quadratic and positive definite in the deviation from a state with a uniform flow, made up of the internal and 'external' potential energies (due to the displacement of the isopycnals and the interface) and the kinetic energy.

Eady's model of baroclinic instability is extended with the present system, i.e. including the effect of the free lower boundary. The integrals of motion give instability conditions that are both necessary and sufficient. If the geostrophic slope of the interface is such that density increases in opposite directions at the top and bottom boundaries, then the basic flow is nonlinearly stable. For very weak internal stratification (as compared with the density jump at the interface) normal modes instability is similar to that of a simpler model, with a rigid but sloping bottom. For stronger stratification, though, the deformation of the lower boundary by the perturbation field also plays an important role, as shown in the dispersion relation, the structure of growing perturbations, and the energetics of the instability. The energy of long growing perturbations is mostly internal potential, whereas short ones have an important fraction of kinetic energy and, for strong enough stratification, external potential.

1. Introduction

Eady (1949) showed how the potential energy associated with the geostrophic isopycnal tilt can be used to generate variability in scales not present in any forcing, through the process of baroclinic instability. Eady used the simplest possible setting: a quasi-geostrophic model with uniform stratification N^2 and constant basic shear $\partial_z U$ as well as rigid horizontal boundaries. Eady's model is not appropriate for the real ocean, where most variability is concentrated near the surface. One possible improvement consists in using non-uniform $N^2(z)$ and $U'(z)$, as done by Gill, Green & Simmons (1974). Another possibility, somewhat simpler, is to consider a shallow active layer floating on top of a denser and (assumed) motionless fluid. This 'reduced gravity' structure is admittedly only a crude representation of the ocean which, nevertheless, has been used extensively with significant success. The f-plane baroclinic instability problem has been reformulated with reduced-gravity models, but assuming a limited vertical structure, by Young (1994), Young & Chen

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(1995), and Ripa (1995). On the other hand, Fukamachi, McCreary & Proehl (1995) and Beron-Vera & Ripa (1997) discuss the linear instability problem making no assumptions on the vertical structure of the perturbations. A variable Coriolis parameter (β-effect) was considered by Ripa (1999*b*), assuming that the perturbation streamfunction has a linear structure with depth, and by Olascoaga & Ripa (1999) with a $2\frac{1}{2}$-layer model, i.e. allowing for a free lower boundary in the problem of Phillips (1954). In Ripa (1999*b*) there is a table showing the different regions of parameter space covered by the works mentioned above as well as the present one; Pierrehumbert & Swanson (1995) give a general review of the subject of baroclinic instability. Both a free boundary and $\beta \neq 0$ introduce important differences with the results of Eady (1949), in particular larger growth rates and the interplay of two horizontal length scales. The motivation for our papers is to understand the physical origin of these differences. Since the instabilities found by Fukamachi *et al.* (1995) and Beron-Vera & Ripa (1997) are of low frequency, a quasi-geostrophic model with a free boundary is developed here, a system that simplifies the study of the nonlinear aspects of the problem (such as the energetics of the instability and saturation of perturbations). A layered quasi-geostrophic model with a reduced-gravity lower boundary condition is presented in Ripa (1992*c*) – and used by Olascoaga & Ripa (1999), but in that system each layer has uniform density. The model developed here, on the other hand, not only has an arbitrary vertical structure but it also supports non-isopycnal upper and lower boundaries, allowing for the future inclusion of buoyancy flux through the model boundaries, which is a crucial component of mixed-layer dynamics and ocean–atmosphere interactions (Haine & Marshall 1998). Achterberg & Ingersoll (1989) have used a similar model (with a different lower boundary condition, though) to study Jupiter's atmosphere.

Model equations are developed in § 2; the only difference with the classical three-dimensional quasi-geostrophic model, e.g. that of Gill *et al.* (1974), is the presence of the freely moving interface, which separates the active and passive layers. The Hamiltonian structure of the model, shown in this section, is used in Ripa (1999*a*) for the derivation of a few-component approximation, using the method of Meacham, Morrison & Flierl (1997).

The generalization of Eady's baroclinic instability problem, with the addition of a free lower boundary and in the f-plane, is addressed first in § 3 using the integrals of motion to derive sufficient stability conditions. These are similar to those found by Mu *et al.* (1994) for Phillips' problem, namely a short-wave cutoff, function of the parameters of the problem. However, these parameters are different in the two cases: the stratification s and slope v for this paper and Charney's b number for Phillips' model (all three parameters are defined in § 3). In addition, nonlinear stability is easily proved for $v > 1$, whereas finite-amplitude bounds to the growth of both the total and zonal perturbations for the $v < 1$ cases are presented in Ripa (1999*a*); these bounds are satisfied by the few-components model developed in that paper. The analogy with Phillips' problem is stability $|b| > 2$ and the bounds derived by Shepherd (1988, 1993) for $|b| < 2$. In Ripa (1999*b*) and Olascoaga & Ripa (1999) all three parameters (s, v, b) come into play as independent variables.

The normal modes analysis done by Beron-Vera & Ripa (1997) for the f-plane case is extended to the whole parameter space in § 4. The parameters for the onset of spectral instability coincide with those found in § 3 from the integrals of motion. The physical origin of the differences between Eady's problem and the present one are investigated here (e.g. the effect of the interface slope and its rigidity). The linear analysis has been further extended in Ripa (1999*c*), where it is shown, in particular,

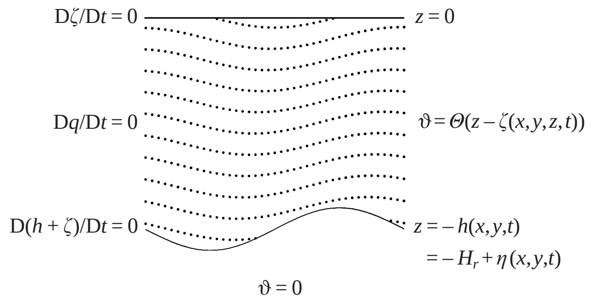

FIGURE 1. Vertical structure of the model. The dotted lines indicate isopycnals. The (constant-density) lower layer is motionless. The definitions of the isopycnal displacement ζ and layer depth h are shown on the right. Model equations are depicted on the left, where q is the potential vorticity.

that the growth rate enhancement produced by the β-effect reported in Ripa (1999*b*) and Olascoaga & Ripa (1999) is not due to the limited vertical resolution of these models but, rather, it is a general result, also present in the continuously stratified model developed here. The modified quasi-geostrophic model derived by White (1977), which incorporates non-Boussinesq effects, is formally similar to the present one at the lower boundary; the differences between the two models are also clarified in this section.

The *a priori* and *linear* analysis of §§ 3 and 4 is completed with the study of small-amplitude *nonlinearities* of the problem in § 5, where the energetics are discussed. Classification of instabilities in terms of energy transfers (between the mean flow and a growing perturbation) is completely misleading with primitive equation models, for which the perturbation energy may have any sign, depending on the variables and coordinates chosen to describe the problem. For instance, in Kelvin–Helmholtz instability the energy of a growing perturbation is positive if depth is used as an independent variable or zero if, instead, density is the vertical coordinate (Ripa 1990). However, this is not the case for quasi-geostrophic theory, for which the difference between the two types of vertical coordinates is, by assumption, irrelevant, and perturbation energy is positive definite. The form of a growing wave and its rectification, discussed in this section, led to the formulation of the minimal nonlinear model reported in Ripa (1999*a*).

The conclusions are presented in § 6. The Appendix gives the details of the derivation of the streamfunction from the boundary density and interior potential vorticity fields, for the particular case of a channel. Part of the pressure and velocity fields associated with a vanishing along-channel wavenumber k has the form of $k = 0$ Kelvin waves; this is interesting in view of the presence of long Kelvin waves in other balanced models (Kushner, McIntyre & Shepherd 1998).

2. Model equations

The first goal of this paper is to develop a subinertial model of an active layer of fluid (in $-h(\boldsymbol{x}, t) \leqslant z \leqslant 0$) lying on top of a motionless layer ($z < -h(\boldsymbol{x}, t)$) with constant density ρ_{deep} (see figure 1). The structure is similar to that of a $1\frac{1}{2}$-layer or 1-layer reduced-gravity model, except that all dynamical fields (density ρ, velocity $(\boldsymbol{u}, w)$, and kinematic pressure p_{total}) are allowed to have arbitrary vertical (as well as horizontal and temporal) variations within the active layer. Let $\vartheta =$

$g\left(\rho_{deep} - \rho(\boldsymbol{x}, z, t)\right)/\rho_0$ be the buoyancy relative to the passive layer, where ρ_0 is a constant density used in the Boussinesq approximation. Instead of $\vartheta(\boldsymbol{x}, z, t)$ it is convenient to use the vertical displacement $\zeta(\boldsymbol{x}, z, t)$, defined by $\vartheta = \Theta(z - \zeta)$ for $-h \leqslant z \leqslant 0$ (and, of course, $\vartheta = 0$ for $z < -h$), where $\Theta(z)$ is some reference profile such that $N^2(z) := \mathrm{d}\Theta(z)/\mathrm{d}z > 0$. Density conservation $\mathrm{D}\rho/\mathrm{D}t = 0$ is equivalent to $\mathrm{D}(z - \zeta)/\mathrm{D}t = 0$, giving the vertical velocity as $w = \mathrm{D}\zeta/\mathrm{D}t$, which in particular must vanish at the (rigid) upper boundary, $z = 0$. The interface is also a material surface (no entrainment of lower fluid is considered at this time), i.e. $\mathrm{D}(z + h)/\mathrm{D}t = 0$ at $z = -h$. Consequently, exact upper and lower boundary conditions are

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0 \quad \text{at } z = 0, \tag{2.1a}$$

$$\frac{\mathrm{D}(h + \zeta)}{\mathrm{D}t} = 0 \quad \text{at } z = -h. \tag{2.1b}$$

Trivial solutions of these equations are given by an isopycnal surface (ζ = const. at $z = 0$) or interface ($h + \zeta$ = const. at $z = -h$), but the interest here lies in the more general case of non-isopycnal boundaries. The physical interpretation of these equations is that each fluid element, in either of the boundaries, stays there and keeps the value of its density; changes in this property due to buoyancy fluxes through the surface or entrainment of deep water can be incorporated as forcing terms in the right-hand sides.

Isobars are horizontal, by definition, in the lower layer: $p_{total} = -gz\rho_{deep}/\rho_0$ for $z < -h$. Let us write in the active layer $p_{total} = P(z) - gz\rho_{deep}/\rho_0 + p(\boldsymbol{x}, z, t)$, for $-h \leqslant z \leqslant 0$, where $\mathrm{d}P/\mathrm{d}z = \Theta(z)$ and p is the (kinematic) pressure variation field. The hydrostatic balance, $\partial_z p_{total} = -g\rho/\rho_0$, and the pressure continuity condition at the interface give $\partial_z p = \Theta(z - \zeta) - \Theta(z)$ at $-h \leqslant z \leqslant 0$ and $p = -P(-h)$ at $z = -h$. For small isopycnals and interface displacements, say $\zeta \to 0$ and $\eta := H_r - h(\boldsymbol{x}, t) \to 0$, the equations used to evaluate the pressure perturbation field are simplified to

$$\partial_z p = -N^2\zeta, \quad -H_r \leqslant z \leqslant 0, \tag{2.2a}$$

$$p = -g_b\eta, \quad z = -H_r. \tag{2.2b}$$

The symbol g_b denotes the buoyancy at the base of the active layer, namely $g_b := \Theta(-H_r)$, a parameter which plays a fundamental role in this model.

These equations are used in the development of the quasi-geostrophic approximation, for which all deviations from a motionless reference state (with $\zeta = \eta = 0$) are considered $O(\varepsilon)$, where $\varepsilon \to 0$ is a Rossby number; nonlinear advection terms are kept because it is also assumed $\partial_t = O(\varepsilon)$. The lowest order of all dynamical fields is written in terms of a single one, the streamfunction ψ, namely

$$\boldsymbol{u} = \hat{\boldsymbol{z}} \times \nabla\psi, \qquad p = f_0\psi, \qquad \zeta = -f_0 N^{-2}\partial_z\psi, \qquad \eta = -f_0 g_b^{-1}\psi_- \tag{2.3a–d}$$

$+ O(\varepsilon^2)$, where the subscripts + and − mean 'evaluated (or defined) at $z = 0$ and $z = -H_r$', respectively. (From now on, the symbols ($\boldsymbol{u}$, p, ζ, η) denote the $O(\varepsilon)$ geostrophic fields.) Equation (2.3a) is a consequence of the vanishing divergence of the horizontal velocity field (since $w = O(\varepsilon^2)$) and, together with (2.3b), gives the geostrophic balance; (2.3c) gives the hydrostatic balance (2.2a); finally, (2.3d) gives the boundary condition at the interface (2.2b).

Prognostic fields are the quasi-geostrophic potential vorticity†

$$\boxed{q\,(\boldsymbol{x},z,t) := f + \hat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{u} - f_0\partial_z\zeta = f + \nabla^2\psi + \partial_z(f_0^2N^{-2}\partial_z\psi),} \tag{2.4}$$

and the buoyancy changes at the surface and the interface, proportional to

$$\boxed{\begin{aligned} q_+\,(\boldsymbol{x},t) &:= f_0H_r^{-1}\zeta_+ = -f_0^2H_r^{-1}\left(N^{-2}\partial_z\psi\right)_+,\\ q_-\,(\boldsymbol{x},t) &:= -f_0H_r^{-1}\,(\zeta_- - \eta) = f_0^2H_r^{-1}\left(N^{-2}\partial_z\psi - g_b^{-1}\psi\right)_-. \end{aligned}} \tag{2.5}$$

From now on, q_n will denote either $q\,(\boldsymbol{x},\,z,\,t)$, $q_+\,(\boldsymbol{x},\,t)$, or $q_-\,(\boldsymbol{x},\,t)$, in their respective domains (namely, $\boldsymbol{x}\in D$ for all q_n, and $-H_r < z < 0$ for $q_n = q$, $z = 0$ for $q_n = q_+$, or $z = -H_r$ for $q_n = q_-$); a similar notation will be used for $\psi_n = \psi$, ψ_+, or ψ_-. The evolution equations are conservation of potential vorticity in the interior and of density at the upper and lower boundaries, equations (2.1), which take the form

$$\boxed{\partial_t q_n + [\psi_n, q_n] = 0,} \tag{2.6}$$

with $[A,B] = \hat{\boldsymbol{z}}\cdot\nabla A\times\nabla B$ the horizontal Jacobian. (Of course, $\partial_t A + [\psi, A]$ is the $O(\varepsilon)$ part of the material derivative $\partial_t A + \boldsymbol{u}\cdot\nabla A + w\partial_z A$.) This system is complemented with the conditions of constancy of Kelvin circulations $\oint_{\partial D_i}\nabla\psi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}l = \gamma_i(z)$ ($-H_r < z < 0$; ∂D is the union of the disconnected coasts ∂D_i), and of vanishing of the normal velocity at the coasts, $\hat{\boldsymbol{z}}\times\nabla\psi\cdot\hat{\boldsymbol{n}} = 0$ ($\boldsymbol{x}\in\partial D$, $-H_r < z < 0$). It can be shown that $\psi(\boldsymbol{x},\,z,\,t)$ can be uniquely calculated from $q(\boldsymbol{x},\,z,\,t)$, $q_\pm(\boldsymbol{x},\,t)$, and $\gamma_i(z)$, at any time, and therefore this is a well posed problem. This inversion is explicitly developed in the Appendix for the particular case of a channel.

The Coriolis parameter and nabla operator will be represented as $f = f_0 + \beta y$ and $\nabla = \left(\partial_x,\,\partial_y\right)$ even though in a correct β-plane approximation some geometric terms must be included (Ripa 1997). Thus if longitude and latitude are linear functions of x and y, respectively, the squared horizontal arc element is $\gamma^2\mathrm{d}x^2 + \mathrm{d}y^2$, where $\gamma = 1 - \tan\vartheta_0\,y/R_T + O(y^2)$, with ϑ_0 the reference latitude and R_T is the mean radius of the Earth. Assuming $y/R_T = O(\varepsilon)$, however, it is found that these geometric (non-Cartesian) terms make a higher-order (in ε) contribution to variable definitions and model equations (Pedlosky 1979; Ripa 1997), and are conveniently ignored in (2.3) and other equations, treating ∇ *as if* the geometry were Cartesian.

2.1. *Conservation laws and Hamiltonian structure*

Three integrals of motion are derived from the quasi-geostrophic model (2.6), namely

$$\begin{aligned} \mathscr{E}[\psi] &= \tfrac{1}{2}\langle\lfloor \boldsymbol{u}^2 + N^2\zeta^2\rceil^z + g_bH_r^{-1}\eta^2\rangle \\ &= \tfrac{1}{2}\langle\lfloor(\nabla\psi)^2 + f_0^2N^{-2}(\partial_z\psi)^2\rceil^z + R^{-2}\psi_-^2\rangle \end{aligned} \tag{2.7}$$

is a *free energy*, quadratic and positive definite in the deviation from the state of rest,

$$\mathscr{M}\,[\psi] = \left\langle\lfloor u\rceil^z + f_0yH_r^{-1}\eta\right\rangle = \left\langle\lfloor-\partial_y\psi\rceil^z + R^{-2}y\,\psi_-\right\rangle$$

† The primitive equations version of Ertel's potential vorticity is $q^{PE} = [\hat{\boldsymbol{z}}\times\partial_z\boldsymbol{u} + \hat{\boldsymbol{z}}\,(f + \hat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{u})]\cdot[(\nabla + \hat{\boldsymbol{z}}\,\partial_z)\,F\,(z-\zeta)]$, where $F\,(z-\zeta)$ represents an arbitrary function of density. Now, q equals the expansion of q^{PE} up to linear order in ε *only if* $F\,(z-\zeta) = z-\zeta$. For other choices of $F\,(z-\zeta)$, there is an extra $O\,(\varepsilon)$ term in $q^{PE} - q$ which, nevertheless, cancels out in the evolution equation, i.e. for any $F\,(z-\zeta)$ it is $\mathrm{D}q^{PE}/\mathrm{D}t = \partial_t q + [\psi, q] + O\left(\varepsilon^3\right)$; see also § 6.5 of Pedlosky (1979).

(modulo constant terms) is the *zonal momentum*, including a Coriolis contribution related to changes in the thickness of the active layer, and

$$\mathscr{C}[\psi] = \langle \lfloor \phi(q,z) \rceil^z + \phi_+(q_+) + \phi_-(q_-) \rangle$$

is a general *Casimir*, where the ϕ_n are arbitrary functions. The notation is such that horizontal and vertical averages are denoted by $\langle \cdots \rangle = \iint_D (\cdots) \mathrm{d}x\,\mathrm{d}y / \iint_D \mathrm{d}x\,\mathrm{d}y$ and $\lfloor \cdots \rceil^z = H_r^{-1} \int_{-H_r}^0 (\cdots)\mathrm{d}z$, respectively, and

$$R = \sqrt{g_b H_r} / |f_0|$$

is the external deformation radius. Conservation of $\mathscr{E}$ and $\mathscr{C}$ is guaranteed by the boundary conditions, whereas $\mathscr{M}$ is a constant of motion only when the horizontal domain D is invariant under zonal translations: a zonal channel (with the limiting cases of a half-plane or the whole plane). These conservation laws are not a spurious result of the quasi-geostrophic approximation but, rather, correspond to those of the 'parent' primitive equations model (Ripa 1995).

Notice that since $\mathscr{E}[\psi]$ is exactly a quadratic functional of ψ, it follows that $\mathscr{E}[\psi+\delta\psi] = \mathscr{E}[\psi] + \delta\mathscr{E}[\psi,\delta\psi] + \mathscr{E}[\delta\psi]$, i.e. the second variation is $\delta^2\mathscr{E}[\psi,\delta\psi] = \mathscr{E}[\delta\psi] = O(\delta\psi^2)$ and higher-order variations vanish. More explicitly $\mathscr{E}[\psi+\delta\psi] - \mathscr{E}[\psi] = \langle \lfloor \nabla\cdot(\psi\nabla\delta\psi) + \partial_z\left(f_0^2 N^{-2}\psi\,\partial_z\delta\psi\right) - \psi\delta q \rceil^z + R^{-2}\,\psi_-\delta\psi_- \rangle + \mathscr{E}[\delta\psi]$, where $\delta q = \nabla^2\delta\psi + \partial_z\left(f_0^2 N^{-2}\partial_z\delta\psi\right)$ and the fields ψ and $\delta\psi$ are arbitrary. Integration of the first two terms yields

$$\delta\mathscr{E}[\psi,\delta\psi] = \sum_i A^{-1} \lfloor \psi|_{x\in\partial D_i}\, \delta\gamma_i \rceil^z - \langle \lfloor \psi\,\delta q \rceil^z + \psi_+\delta q_+ + \psi_-\delta q_- \rangle \tag{2.8}$$

for the first variation, where $A = \iint_D \mathrm{d}x\,\mathrm{d}y$ is the area of the horizontal domain. Finally, if this domain is a zonal channel then it also follows that

$$\mathscr{M}[\psi+\delta\psi] - \mathscr{M}[\psi] \equiv \delta\mathscr{M} = \langle y\,\lfloor \delta q \rceil^z + y\delta q_+ + y\delta q_- \rangle - \sum_i A^{-1} y_i \delta\gamma_i.$$

Equation (2.8) suggests as coordinates in state space the fields $q(\boldsymbol{x},z,t)$, $q_+(\boldsymbol{x},t)$, $q_-(\boldsymbol{x},t)$, and $\gamma_i(z)$. Defining the Lie–Poisson bracket between two arbitrary functionals $\mathscr{A}[q_n,\gamma_i]$ and $\mathscr{B}[q_n,\gamma_i]$ by

$$\{\mathscr{A},\mathscr{B}\} := \left\langle \left\lfloor q\left[\frac{\delta\mathscr{A}}{\delta q},\frac{\delta\mathscr{B}}{\delta q}\right]\right\rceil^z + q_+\left[\frac{\delta\mathscr{A}}{\delta q_+},\frac{\delta\mathscr{B}}{\delta q_+}\right] + q_-\left[\frac{\delta\mathscr{A}}{\delta q_-},\frac{\delta\mathscr{B}}{\delta q_-}\right] \right\rangle,$$

it is $(\partial_t + \alpha\partial_x)\,q_n = \{q_n,\mathscr{H}_\alpha\}$, where

$$\mathscr{H}_\alpha[\psi] = \mathscr{E}[\psi] - \alpha\mathscr{M}[\psi] + \mathscr{C}[\psi]. \tag{2.9}$$

Furthermore, the Lie–Poisson bracket is antisymmetric and satisfies the Jacobi identity. Therefore, $\mathscr{H}_\alpha[\psi]$, with arbitrary $\mathscr{C}[\psi]$, is a Hamiltonian for the evolution equations in a frame moving with velocity $\alpha\hat{\boldsymbol{x}}$ with respect to the Earth; the form of the Lie–Poisson bracket is standard for quasi-geostrophic models (see for instance McIntyre & Shepherd 1987; Shepherd 1990; Morrison 1994; Ripa 1994).

3. Formal stability

The integrals of motion of a dynamical system can be used to establish *a priori* stability and instability conditions (Abarbanel *et al.* 1986; Liu, Mu & Shepherd 1996;

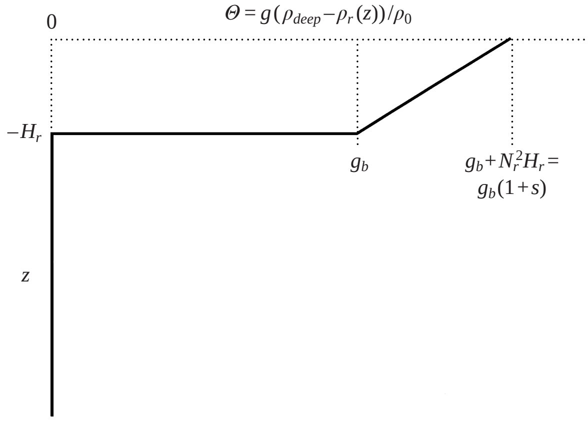

FIGURE 2. Buoyancy (relative to the deep water) in the motionless reference state. The buoyancy jump at the base of the active layer ($z = -H_r$) equals g_b, whereas the buoyancy variation within that layer equals $N_r^2 H_r$ ($= s\, g_b$). The stratification parameter s is used in the following figures.

Ripa 1992*a*, *b*, *c*, 1993). Let the streamfunction be split as $\psi = \Psi + \delta\psi$, where the basic flow Ψ is an exact solution of the model and the perturbation $\delta\psi$ is arbitrary. The evolution equations take the form

$$\delta q_{n,t} + [\Psi_n, \delta q_n] + [\delta\psi_n, Q_n] = -[\delta\psi_n, \delta q_n]\,. \tag{3.1}$$

In order to solve these equations it is necessary to calculate $\delta\psi\,(\boldsymbol{x}, z, t)$ as a functional of $\delta q\,(\boldsymbol{x}, z, t)$, $\delta q_\pm\,(\boldsymbol{x}, t)$, and $\delta\gamma_i\,(z)$, using the procedure developed in the Appendix.

Now, if $\partial_t \Psi = 0$ it may be possible to construct the integral of motion (2.9) where α is an arbitrary constant (which can be different from zero only for a symmetric basic state $\partial_x \Psi = 0$) and the Casimir $\mathscr{C}$ is chosen so that $\delta\mathscr{H}_\alpha\,[\Psi, \delta\psi] = 0$. Thus $\mathscr{H}_\alpha\,[\Psi + \delta\psi] = \mathscr{H}_\alpha\,[\Psi] + \delta\mathscr{H}_\alpha\,[\Psi, \delta\psi] + \delta^2\mathscr{H}_\alpha\,[\Psi, \delta\psi] + \cdots$ where the first term is a constant, by definition, the first variation vanishes by construction, and the second variation equals

$$\delta^2\mathscr{H}_\alpha\,[\Psi, \delta\psi] = \mathscr{E}\,[\delta\psi] + \delta^2\mathscr{C}\,[\Psi, \delta\psi]\,, \tag{3.2a}$$

where

$$\delta^2\mathscr{C}\,[\Psi, \delta\psi] = \frac{1}{2}\left\langle \left\lfloor \frac{\alpha - U}{Q_{,y}}\,(\delta q)^2 \right\rceil^z + \frac{\alpha - U_+}{Q_{+,y}}\,(\delta q_+)^2 + \frac{\alpha - U_-}{Q_{+,y}}\,(\delta q_-)^2 \right\rangle. \tag{3.2b}$$

This second variation $\delta^2\mathscr{H}_\alpha$ is an integral of motion of the linearized equations, obtained by neglecting the nonlinear term in the right-hand side of (3.1), which can be used to obtain stability/instability conditions: If there is a value of α such that $\delta^2\mathscr{H}_\alpha$ is sign definite, then the basic state Ψ is formally stable. On the other hand, if Ψ is unstable then for *any* α there must be perturbations such that $\delta^2\mathscr{H}_\alpha = 0$.

Consider the simplest reduced-gravity problem, namely a reference buoyancy relative to the passive layer

$$\Theta(z) = g_b + N_r^2\,(z + H_r)\,, \tag{3.3}$$

where g_b and N_r^2 are positive constants (see figure 2), and a basic flow, whose stability is studied, of the form

$$U\,(z) = U_b + \left(1 + z/H_r\right) U_s, \tag{3.4}$$

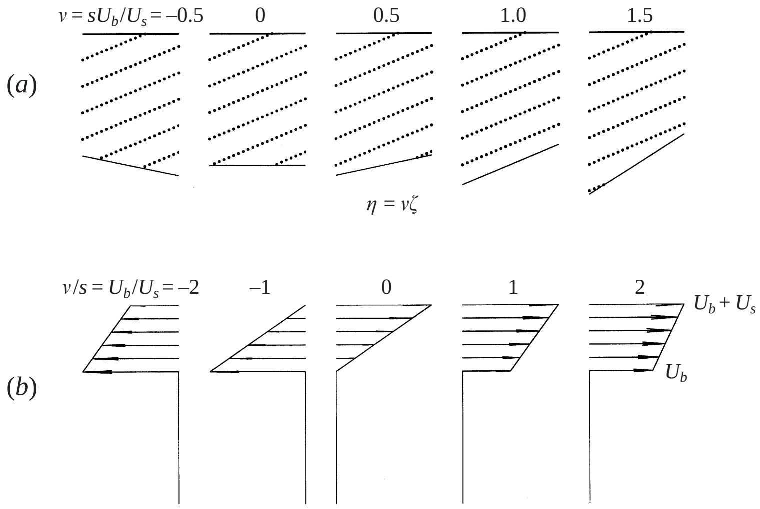

FIGURE 3. Mass (a) and velocity (b) fields at the basic state used to study the baroclinic instability problem. For $v > 0$ (< 0), interface and isopycnals slope in the same (opposite) directions and for $v = 1$ the lower boundary has constant density; this is important for the interpretation of the following figures. The horizontal coordinate is y in (a) and x in (b).

where U_b and U_s are real constants. Four non-dimensional parameters are needed to characterize this baroclinic instability problem

$$s = \frac{N_r^2 H_r}{g_b}, \quad v = s\frac{U_b}{U_s}, \quad b = \frac{\beta L^2}{U_s}, \quad \kappa = |\boldsymbol{k}|\, L, \tag{3.5}$$

where $L = N_r H_r/|f_0|$ is Eady's horizontal scale. The first basic-state parameter, s, is a measure of the stratification within the active layer relative to the buoyancy jump at the interface.† The second one, v, is the geostrophic slope of the interface (proportional to the velocity jump U_b between the base of the active layer and the passive one), relative to the slope of the isopycnals (see figure 3). The third one, Charney number b, is proportional to the meridional gradient β of the Coriolis parameter, normalized so that b/v equals the ratio of the planetary to the topographic contributions to the gradient of the basic-state potential vorticity f/H (Ripa 1999b).

The basic flow (3.4) corresponds to $\Psi = -U_b y - \left(1 + z/H_r\right) U_s y$. Figure 3 shows the basic-state velocity $U(z)$ (part a) and mass $[\rho(y,z), H(y)]$ (part b) fields, for different values of U_b/U_s (which is the equivalent of s^{-1}, for the basic velocity instead of for the reference buoyancy). Notice that the value $U_b/U_s = 0$ separates the cases where the velocity at the bottom of the active layer and the shear have the same or opposite signs or, equivalently, the cases where isopycnals and the interface slope in the same or opposite directions. More precisely the isopycnals and interface elevation fields in the basic state are given by $\zeta = (f_0/N_r^2 H_r) U_s y$ and $\eta = (f_0/g_b) U_b y \equiv v\zeta$. When $v > 1$, density increases in opposite directions at the top and bottom boundaries (see figure 3).

In this case, the three basic fields Q_n are linear functions of y. It turns out that the

† The corresponding parameters used by Ripa (1995) and Beron–Vera & Ripa (1997) are $g_r = g_b\left(1 + s/2\right)$, $S = s/(2+s)$, $\overline{U} = U_b + \frac{1}{2}U_s$, and $U_\sigma = \frac{1}{2}U_s$; the variables chosen here simplify the notation. Notice that $0 < S < 1$ but $0 < s < \infty$.

integral of motion $\mathscr{H}_\alpha[\psi]$ is exactly quadratic in the perturbation, $\mathscr{H}_\alpha[\Psi+\delta\psi]-\mathscr{H}_\alpha[\Psi]\equiv\delta^2\mathscr{H}_\alpha[\Psi,\delta\psi]$ from (3.2*a*). The pseudomomentum $-\partial_\alpha\left(\delta^2\mathscr{H}_\alpha[\Psi,\delta\psi]\right)$ is but a linear combination of the variances of the prognostic variables δq_n. The relative sign of the coefficients $Q_{n,y}^{-1}$, where

$$\left[Q_{,y},\, Q_{+,y},\, Q_{-,y}\right]=L^{-2}U_s\,[b,\,1,\,v-1]\,,$$

gives *a priori* information on the stability/instability properties of the basic flow, namely if the three coefficients have the same sign then the basic flow is stable; moreover, it is easy to prove nonlinear stability in suitable defined norms (Ripa 1999*c*). On the other hand, if the basic flow is unstable, then one of the coefficients must have sign opposite to that of the other two, and therefore for a growing perturbation the variance of the corresponding δq_n must be important, because it balances the other two terms. This *a priori* information on the structure of growing perturbations was used in the problem of vortex stability in a two-layer model in Ripa (1992*a*). The analysis of the relative signs of the coefficients $Q_{n,y}^{-1}$ yields the following results:

$$\begin{array}{c|cc} & v<1 & v>1\\ \hline b>0 & \delta q_- & \text{stable}\\ b<0 & \delta q_+ & \delta q\end{array}\tag{3.6}$$

where each entry indicates either stability of a basic flow with those parameters or which one of the three fields δq_n must be an important part of a growing perturbation (if the basic flow is unstable).

When $\beta=0$ it is not possible to construct an integral of motion $\mathscr{H}_\alpha[\delta\psi]$ whose first variation vanishes for arbitrary perturbations (which include $\delta q\neq 0$). This complicates very much the derivation of normed stability theorems (Liu *et al.* 1996) but not in the case of formal stability. In fact, for $\beta=0$ it follows that $\delta q_{,t}+[\Psi+\delta\psi,\delta q]=0$ and then $\delta q=0$ is a constraint which is maintained by the dynamics. The integral $\mathscr{H}_\alpha[\delta\psi]$ (3.2*a*) without the first term in $\delta^2\mathscr{C}$ (3.2*b*) is a finite-amplitude constant of motion of the constrained dynamics. This might seem surprising, because β appears in the denominator of the suppressed term. However, this term is $\frac{1}{2}\beta^{-1}\left\langle\left\lfloor(\alpha-U)\,\delta q^2\right\rceil^z\right\rangle=\frac{1}{2}\beta\left\langle\left\lfloor(\alpha-U)\,\delta Y^2\right\rceil^z\right\rangle$, where $Y\,(x,z;q)$ is the y-coordinate of a q isoline, and thus this term does tend to zero as $\beta\to 0$ if δY is bounded.

For $\beta=0$ and $\delta q=0$, the last column in (3.6) shows that the basic flow is stable for $v>1$, independently of the sign of U_s, because the reduced $\mathscr{H}_\alpha[\delta\psi]_{\delta q=0}$ is positive definite; this is known as Arnold's first theorem, see figure 4. It is possible to prove stability for some cases with $v<1$, by showing that $\mathscr{H}_\alpha[\delta\psi]$ is negative definite (Arnold's second theorem), but that requires an explicit development of the dynamics in the $\delta q=0$ subspace. This is done using $\delta q_\pm(\boldsymbol{x},t)$ as coordinates or elementary fields; making a Fourier representation of them

$$\delta\boldsymbol{q}\,(\boldsymbol{x},t)=\sum_{k,l}\boldsymbol{q}^{k,l}\,(t)\,\mathrm{e}^{\mathrm{i}kx}\sin ly,$$

equation (A 4) shows how to calculate $\delta\psi\,(\boldsymbol{x},z,t)$ from $\delta q_\pm(\boldsymbol{x},t)$. A bold face symbol indicates a vector field with top and bottom components, e.g. $\delta\boldsymbol{q}=(\delta q_+\ \delta q_-)^{\mathrm{T}}$. Evaluating expression (A 4) at $z_+=0$ and $z_-=-H_r$ gives $\delta\psi_\pm(\boldsymbol{x},t)=\delta\psi\left(\boldsymbol{x},z_\pm,t\right)$, which is all that is needed to solve the evolution equations, as

$$\delta\boldsymbol{\psi}\,(\boldsymbol{x},t)=-L^2\sum_{k,l}\mathbb{E}\,\boldsymbol{q}^{k,l}\,(t)\,\mathrm{e}^{\mathrm{i}kx}\sin ly\tag{3.7a}$$

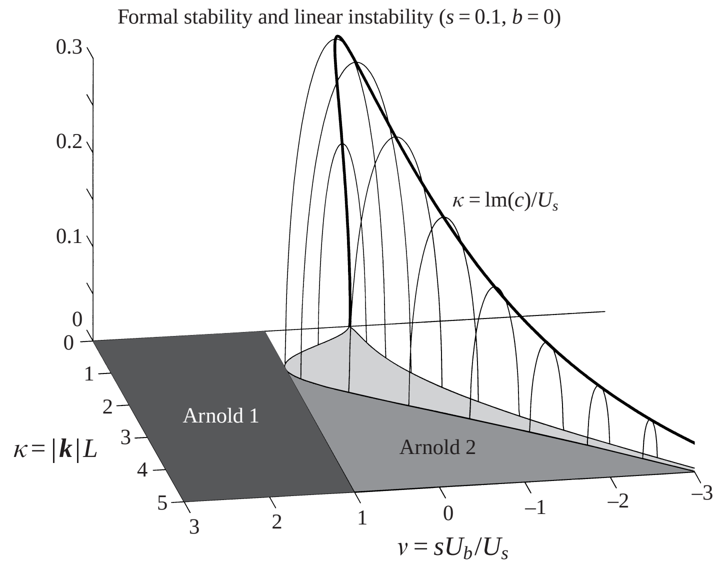

FIGURE 4. Normal mode growth rate $\kappa\, \mathrm{Im}\, c/U_s$ as a function of the wavenumber κ and the interface slope ν, for fixed stratification s and $\beta = 0$. The shaded regions in (ν, κ)-space labelled Arnold 1 and 2 are proved to be stable because the Hamiltonian is positive or negative definite, respectively; in the second case, κ must be interpreted as the minimum wavenumber allowed by the channel.

where

$$\mathbb{E}(\kappa) = \frac{1}{\tau\kappa + s} \begin{pmatrix} 1 + s\tau/\kappa & \sqrt{1-\tau^2} \\ \sqrt{1-\tau^2} & 1 \end{pmatrix} \tag{3.7b}$$

with $\kappa^2 = \left(k^2 + l^2\right) L^2$ and $\tau = \tanh \kappa$. Thus, for a single Fourier component $\delta\boldsymbol{\psi} = -L^2 \mathbb{E} \delta\boldsymbol{q}$ and the matrix $\mathbb{E}$ gives the energy in $\delta\boldsymbol{q}$ space, namely, $\lfloor(\kappa\delta\psi)^2 + (H_r \delta\psi_{,z})^2\rceil^z + s(\delta\psi_-)^2 = L^4(\delta\boldsymbol{q}^{\mathrm{T}} \mathbb{E} \delta\boldsymbol{q})$, in accordance with (2.8). Nonlinear dynamics on the $\delta q = 0$ subspace conserve the quadratic integral, $\mathscr{H}_\alpha[\psi] = \frac{1}{4} L^2 \sum_{k,l} \mathrm{Re}\, [\boldsymbol{q}^{k,l}(t)^\dagger\ \mathbb{H}(\alpha, \kappa) \boldsymbol{q}^{k,l}(t)]$, where

$$\mathbb{H}(\alpha, \kappa) = \mathbb{E}(\kappa) + \begin{pmatrix} a - 1 & 0 \\ 0 & \dfrac{a}{\nu - 1} \end{pmatrix} \tag{3.8}$$

and $a = (\alpha - U_b)/U_s$ is arbitrary.

Two clarifications must be made now. First, in order to calculate $\delta\psi(\boldsymbol{x}, z, t)$, and (3.7$a$) in particular, it was assumed that $\delta\gamma_i(z) = 0$ for simplicity, since a perturbation of the Kelvin circulations only produces a change in the basic flow; see (A 7). Secondly, expression (3.7b) for the matrix $\mathbb{E}$ is not valid when $k = 0$; see (A 6). However, it can be shown that a growing perturbation must have $k \neq 0$.† Allowed wavenumbers κ for a growing perturbation are then

$$\kappa > \kappa_{\min} = \pi L/W \tag{3.9}$$

† Formal stability is guaranteed by an extremum of $\mathscr{E} - \alpha\mathscr{M}$ on the isovortical sheet $\mathscr{C}[\phi_n] = \mathrm{const.}$ for all $\phi_n = \phi_n(q_n)$ (Morrison 1994). The condition $\iint \phi_n(Q_n(y) + \delta q_n) = \mathrm{const.}$ implies $\iint \phi_n'(Q_n(y))\, \delta q_n = 0$ when δq_n is an infinitesimal growing perturbation. Since $Q_n(y)$ is monotonic, choosing $\phi_n(q_n) = 1$ if $q_n > Q_n(y_0)$ and $\phi_n(q_n) = 0$ otherwise, it follows that $\iint \delta(y - y_0)\, \delta q_n = 0$ $\forall y_0$, which in the case of a growing normal mode implies $k \neq 0$ (Ripa 1993).

if the channel is infinite or $\kappa_{\min}^2 = (\pi L/W)^2 + (2\pi L/L_x)^2$ if the channel is periodic, with length L_x.

Formal stability/instability conditions are then derived as follows:

If $\exists\,\alpha$ *such that* $\det \mathbb{H}(\alpha,\kappa) > 0, \forall\kappa \geqslant \kappa_{\min} \Rightarrow$ **STABLE**.

If **UNSTABLE** $\Rightarrow \exists\kappa \geqslant \kappa_{\min}$ *such that* $\det \mathbb{H}(\alpha,\kappa) \leqslant 0, \forall\alpha$.

Now, $\det \mathbb{H}(\alpha,\kappa)$ is a second-order polynomial in α, such that the coefficient of α^2 has the sign of $v-1$. Consequently there is stability for $v > 1$, whereas instability for $v < 1$ and wavenumber κ requires $\max_\alpha \det \mathbb{H}(\alpha,\kappa) < 0$, namely

$$\left.\begin{aligned} \partial_\alpha \det \mathbb{H}(\alpha,\kappa) = 0 \\ \det \mathbb{H}(\alpha,\kappa) = 0 \end{aligned}\right\} \Rightarrow v = v_\pm(s,\kappa),$$

where

$$v_\pm = 2\tau^2 - \kappa\tau + s\left(\tau/\kappa - 1\right) \pm \sqrt{4\left(1-\tau^2\right)(\kappa-\tau)\left(\tau + s/\kappa\right)}. \tag{3.10}$$

These expressions give two values of κ for each $v < 1$, say $\kappa = \kappa_S(v, s)$ and $\kappa = \kappa_L(v, s)$, such that $\det \mathbb{H}(\alpha,\kappa) \leqslant 0$ for any α and $\kappa_L \leqslant \kappa \leqslant \kappa_S$. Figure 4 shows the results for the sign of $\det \mathbb{H}$ in the (v,κ)-space. In the region labelled 'Arnold 1' ($v > 1$) $\det \mathbb{H}(\alpha,\kappa) > 0$ and $\operatorname{tr} \mathbb{H}(\alpha,\kappa) > 0$ in a neighbourhood of $\alpha = \infty$, and therefore the basic flow is nonlinearly stable (the Hamiltonian is positive definite). In the region labelled 'Arnold 2' ($\kappa > \kappa_S(v,s)$) $\det \mathbb{H}(\alpha,\kappa) > 0$ and $\operatorname{tr} \mathbb{H}(\alpha,\kappa) < 0$ in a neighbourhood of the value of α where $\partial_\alpha \det \mathbb{H}(\alpha,\kappa) = 0$; if the minimum wavenumber in the channel is $\kappa_{\min} > \kappa_S$ there is also formal stability, but now because the Hamiltonian is negative definite. Finally, it is shown below that for $\kappa < \kappa_L$ there is spectral (normal mode) stability for that particular wavenumber, but there may be instability if a higher allowed wavenumber falls in the region $\kappa_L \leqslant \kappa \leqslant \kappa_S$.

A similar analysis can be done with the layered quasi-geostrophic model (Paret & Vanneste 1996); in particular, with two layers (the classical Phillips problem) the results of Mu *et al.* (1994) are obtained. In fact, the pseudomomentum of this problem and that of Phillips' model, between horizontal and rigid boundaries, are formally equivalent making the change of parameters $v \longmapsto 2b/(b+2)$: 'Arnold 1' nonlinear stability condition $v > 1$ is equivalent to $|b| > 2$, and the wave enstrophy bounds derived by Shepherd (1988) can be translated to the present problem (Ripa 1999*a*). 'Arnold 2' stability results cannot be quantitatively translated from one problem to the other, essentially because the corresponding energy matrices $\mathbb{E}(\kappa)$ are different. Olascoaga & Ripa (1999) study the layered equivalence of the present problem ($2\frac{1}{2}$-layers, i.e. Phillips' model with a free boundary), for which v and b are independent parameters, with results qualitatively similar to those obtained here: 'Arnold 1' nonlinear stability condition for $|v+b|\,/\,(2-v)$ negative or greater than 1, and 'Arnold 2' condition $\kappa_{\min} > \kappa_S(b,v,s)$, where κ_S coincides with the short-wave cutoff for normal modes instability. The approximate two-field model developed in Ripa (1999*b*) gives a reasonable representation of the spectral instability in (κ,b,v,s) space; however, it lacks enough Casimir integrals of motion to build the equivalent of $\mathscr{H}_\alpha$ and prove formal stability.

4. Linear instability for $b = 0$

An infinitesimal perturbation $\delta q_\pm = a\,\hat{q}_\pm\, e^{ik(x-ct)} \sin ly + O\left(a^2\right)$ satisfies $ik\left(U_\pm - c\right)\hat{q}_\pm + ikQ_{\pm,y}\hat{\varphi}_\pm = 0$. Using $\hat{\boldsymbol{\psi}} = -L^2\mathbb{E}\hat{\boldsymbol{q}}$ from (3.7*b*), the normal mode equations

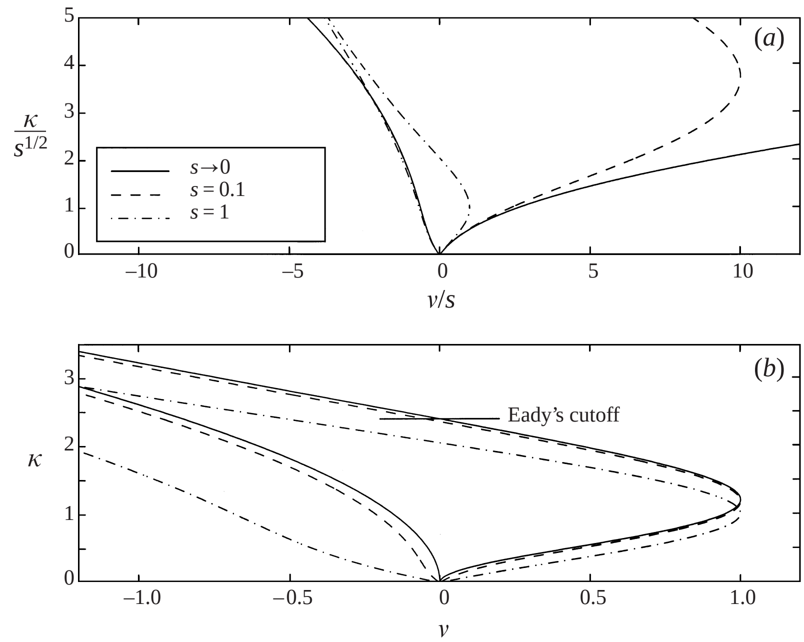

FIGURE 5. Dependence of the instability region on the stratification parameter *s*, using the scaling appropriate for (*a*) long and (*b*) short perturbations when $s \to 0$. The horizontal line shows the short-wave cutoff of Eady's problem (with horizontal and rigid boundaries).

$\mathbb{H}(c,\kappa)\,\hat{\boldsymbol{q}} = \mathbf{0}$ follow, whose non-trivial solutions imply the dispersion relation

$$\frac{c - U_b}{U_s} = \frac{1}{2} - \frac{v + s\tau/\kappa \pm \sqrt{\Delta}}{2\,(s + \kappa\tau)}, \tag{4.1}$$

first obtained by Beron-Vera & Ripa (1997). The discriminant can be written as $\Delta = (v - v_+)\,(v - v_-)$, where $v_\pm\,(s,\kappa)$ from (3.10) give the boundary of negative definite Hamiltonians. Consequently, the spectral instability condition $\Delta < 0$ coincides with the criteria derived from the conservation laws (figure 4), which are then not only necessary but also sufficient. The fact that unstable states are 'surrounded' by formally stable states, with a sign-definite integral of motion, is used for the determination of perturbation saturation bounds (Ripa 1999*a*).

Figure 5 depicts the instability region for different values of the stratification *s*. Figure 5(*b*) shows that, using as variables *v* and *κ*, a value of $s = 0.1$ is very close to the asymptotic curves for $s \to 0$; these asymptotes are obtained by making $s = 0$ in (4.1). The long-perturbations limit (figure 5*a*, for which the appropriate variables are $U_b/U_s = v/s$ and $|\boldsymbol{k}|\,R = \kappa/\sqrt{s}$) was found to depart considerably from Eady's problem by Fukamachi *et al.* (1995), for $v = 0$, and Beron-Vera & Ripa (1997), for $v = O\,(s)$. Inspection of figures 5(*a*) and 5(*b*) shows that Eady's model is a valid approximation of the present problem, with a free boundary, only for $s \to 0$, with $\kappa = O\,(1)$ and $v \ll 1$.

4.1. *Eigenvalues*

Eady's model (which has rigid and horizontal boundaries) presents instability for any U_b, any $U_s \neq 0$, and the wavenumbers $0 < \kappa < \kappa_E = 2.3993573\ldots$ ($\kappa_E \tanh\left(\kappa_E/2\right) = 2$). The present model has both long- and short-perturbation cutoffs, $\kappa_L\,(v,s)$ and

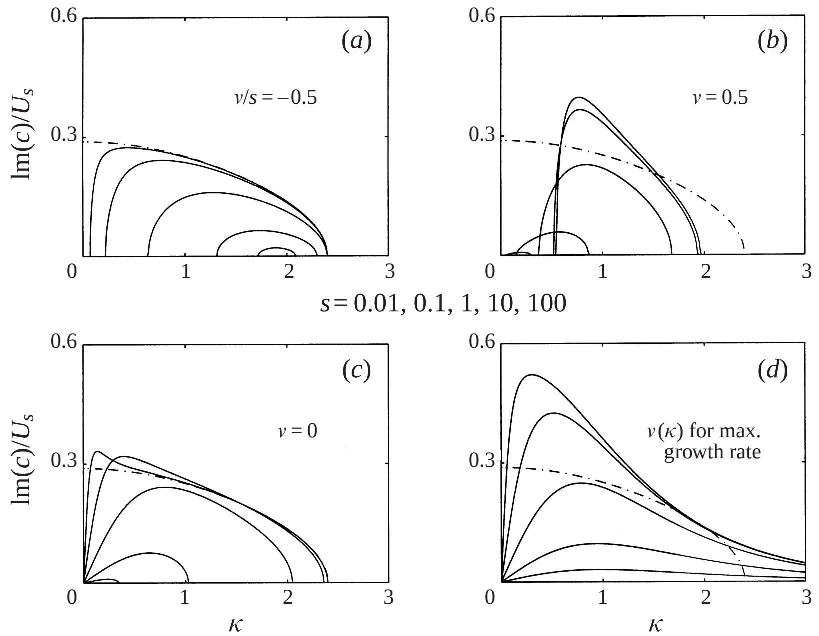

FIGURE 6. Imaginary part of the eigenvalue c (scaled by the shear U_s), as a function of κ and for different values of s and v. The dot-dashed curve corresponds to the result of Eady's model (with rigid horizontal boundaries).

$\kappa_S(v, s)$. These coincide at the maximum value of v, namely

$$v = 1 \Rightarrow \kappa_L = \kappa_S, \quad \kappa_S \frac{1 - \kappa_S \tanh \kappa_S}{\kappa_S - \tanh \kappa_S} = s, \tag{4.2}$$

whereas for $U_b = 0$, the value chosen by Fukamachi *et al.* (1995),

$$v = 0 \Rightarrow \kappa_L = 0, \quad \kappa_S \frac{\kappa_S \sinh \kappa_S - 2\cosh \kappa_S - 2}{\sinh \kappa_S - \kappa \cosh \kappa_S} = s.$$

Notice that for $\kappa \to \infty$, $v_\pm \sim -\kappa \pm 2\sqrt{\kappa}$, i.e. $\kappa_S \sim -v + 2\sqrt{-v}$ and $\kappa_L \sim -v - 2\sqrt{-v}$ as $v \to -\infty$: it is possible to have instability for arbitrarily short perturbations. This peculiarity is related to the choice of an horizontal and rigid top (Ripa 1999*c*); pseudomomentum conservation implies that a topography at that boundary (e.g. the geostrophic slope it would acquire if it were also free) renders the range of unstable v finite.

Both $\kappa_L(v, s)$ and $\kappa_S(v, s)$ decrease with increasing s. For instance, for weak stratifications, $s \to 0$, $\kappa_S(1, s) \uparrow \kappa_E/2$ and $\kappa_S(0, s) \uparrow \kappa_E$. On the other hand, for strong stratifications the dispersion relation becomes

$$s \to \infty : \begin{cases} \kappa = O(s^{-1/2}) \\ c \sim \frac{1}{6} s^{-1} U_s (s\kappa^2 - 3v \pm \sqrt{(s\kappa^2 - 6 + 3v)^2 + 36(v-1)}), \end{cases}$$

i.e. the instability region is shrunk into low wavenumbers, $\kappa_L = |\sqrt{3/s} - \sqrt{3(1-v)/s}|$ and $\kappa_S = \sqrt{3/s} + \sqrt{3(1-v)/s}$, a weak bottom flow is needed for instability, $U_b = O(s^{-1} U_s)$, and the growth rate decreases as $\kappa \operatorname{Im} c/L = O(s^{-3/2} U_s/L)$.

Figure 6 shows $\operatorname{Im} c/U_s$, as a function of κ and for different s. In figure 6(*a*),

$U_b/U_s = -0.5$, a long-perturbations cutoff is seen which tends to zero as $s \to 0$. In figure 6(*c*), $U_b/U_s = 0$, there is no long-perturbations cutoff, but the 'boundary layer' with the scaling $\kappa = O(\sqrt{s})$ is seen, where the solution has not yet asymptoted to the result of Eady's model. For $v = O(1)$, graphs at the right, the results differ completely from those of Eady's model. The long-perturbations cutoff increases with s (upper graphs). Finally, recall that the short-perturbations cutoff can in this model be as large as desired: this is seen in figure 6(*d*), where $\mathrm{Im}\, c/U_s$ is plotted choosing $v = 2\tau^2 - \kappa\tau + s\left(\tau/\kappa - 1\right)$, the mean of the two terms in (3.10), which gives the maximum value of $\mathrm{Im}\, c/U_s$, at each κ.

An interesting result depicted by the curves in figure 6(*d*) is that the asymptotic maximum value of $\mathrm{Im}\, c/U_s$ is twice as large as that from Eady's model, namely $\mathrm{Im}\, c/U_s = (1/\sqrt{3})\,(1 - \epsilon + O\,(s))$, where $\epsilon = \sqrt{15s/16}$. This maximum is reached at $\kappa = \sqrt{\epsilon} + O\left(s^{3/4}\right)$ and $v = \epsilon - \epsilon^2 + O\left(s^{3/2}\right)$; these are values in the 'matching layer' between the scales of short perturbations [$\kappa = O\,(1)$ and $v = O\,(1)$] and long ones [$\kappa = O\left(\sqrt{s}\right)$ and $v = O\,(s)$]. The growth rate of the perturbation $k\,\mathrm{Im}\, c$ not only depends on the value of κ (as $\mathrm{Im}\, c$ does) but also on the width of the channel or, equivalently, on $\kappa_{\min}$ defined in (3.9). Thus, $k\,\mathrm{Im}\, c/\left(U_s/L\right) = \sqrt{\kappa^2 - \kappa_{\min}^2}\,\mathrm{Im}\, c/U_s$ is bounded by $\kappa\,\mathrm{Im}\, c/U_s$, shown in figure 4 for a particular value of s. Making $s \to 0$, for example, the maximum non-dimensional growth rate is found to be

$$\max_{v,\kappa}\left(\kappa\,\mathrm{Im}\, c/U_s\right)_{b=0} \sim 0.367$$

and is reached at $\kappa \sim 1.138$ and $v \sim 0.398$ (see figure 4). This growth rate is about 18% larger than that obtained in Eady's model and 27% smaller than the value predicted in Ripa (1999*b*) and Olascoaga & Ripa (1999) for $b \neq 0$, namely $\max_{v,\kappa,b}\left(\kappa\,\mathrm{Im}\, c/U_s\right) \sim \frac{1}{2}$. One important point is that the maximum growth rate for $b = 0$ is reached at short perturbations, $\kappa = O\,(1)$, whereas for $b \neq 0$ it is reached at intermediate scales, $\kappa = O\left(\sqrt[4]{s}\right)$.

4.2. *Eigenfunctions*

The normal mode equations $\mathbb{H}\,(c,\kappa)\,\hat{\boldsymbol{q}} = \boldsymbol{0}$ give the dispersion relation (4.1) and the eigenfunction

$$\begin{pmatrix} \hat{q}_+ \\ \hat{q}_- \end{pmatrix} \propto \begin{pmatrix} -\sqrt{1-\tau^2}\,\kappa U_s \\ U_s\,(\kappa + s\tau) + \kappa\,(s + \kappa\tau)\,(c - U_b - U_s) \end{pmatrix},$$

which corresponds to $\hat{\psi}\,(z) \propto (c - U_b - U_s)\,\kappa \cosh \kappa\, z/H_r - U_s \sinh \kappa\, z/H_r$. Figure 7 shows some representative plots of the eigenfunctions of the instability. Even though these profiles of amplitude and phase show quite a bit of structure, those of the real and imaginary parts of $\hat{\psi}\,(z)$ (not shown) can be reasonably fitted to straight lines in z, which partially explains the success of the approximate model developed in Ripa (1999*b*) by restricting the streamfunction to have a linear z structure. The variation due to the slope v can be appreciated by comparing rows (*b*) and (*c*). Notice the importance of the stratification, except for the $s \ll \kappa^2$ cases. Only the $v = 0$ and $s = 0$ curve shows the vertical symmetry characteristic of Eady's problem.

4.3. *Effects of interface shape and rigidity*

It is important to try to understand the physical origin of the differences between Eady's problem (with rigid horizontal boundaries) and the present one, which has

the effect of the *basic flow* on the lower boundary, making $H = H_r - f_0 g_b^{-1} U_b y$;

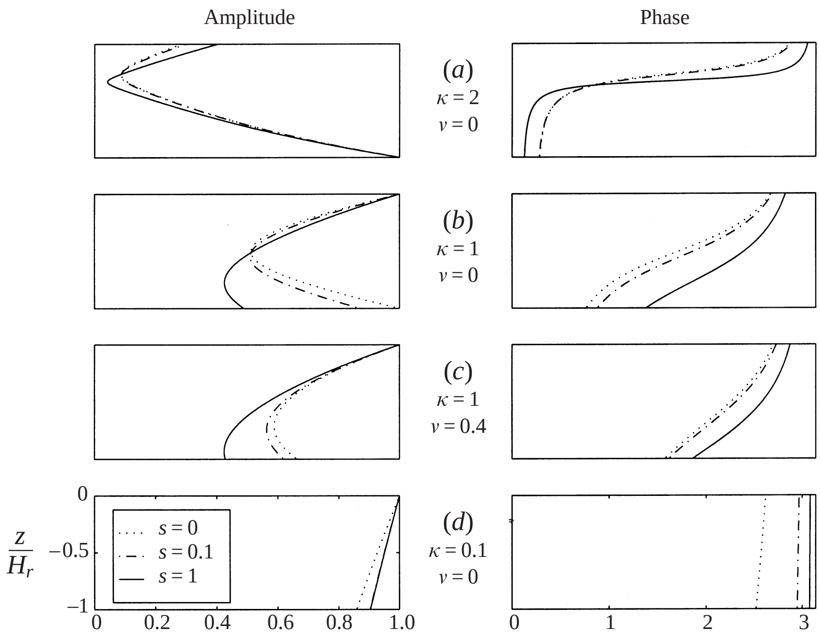

FIGURE 7. Vertical structure of some growing perturbations. (The scale of the amplitude and the origin of the phase are, of course, arbitrary.)

the effect of the *perturbation*, further making $h = H_r - f_0 g_b^{-1} \left(U_b y - \psi'_- \right)$.

Both effects contribute to the solution for the perturbation through the linearized form of the lower boundary equation $\mathrm{D}(h + \zeta)/\mathrm{D}t = 0$. In order to isolate the first effect, we for the moment replace this equation by $\mathrm{D}(H_r - ay + \zeta)/\mathrm{D}t = 0$, which represents a fixed bottom topography $z = -H_r + ay$. This modification gives the dispersion relation derived by Blumsack & Gierasch (1972), which coincides with (4.1) for $s = 0$, κ finite, and $v = aN_r L/U_s$ (so that the basic bottom slope is the same in both cases). The eigenfunctions of the problem with a rigid sloping bottom can be similarly obtained from those of the present problem, making $s \to 0$ while keeping (κ, v) fixed. The importance of the softness of the interface can then be accessed by comparing the $s = 0$ and $s \neq 0$ curves in figure 5 for the instability regions, figure 6 for the eigenvalues, and figure 7 for the eigenfunctions. It is clear that for $s \ll \kappa^2$ the differences between Eady's problem and the present model for normal modes instability are solely due to the sloping of the interface produced by the basic flow (seen for $|v| \gtrsim s$), whereas for $s \gtrsim \kappa^2$ the deformation of the interface produced by the perturbation is also important. This point is reinforced in § 5, where incipient nonlinearities are considered.

4.4. *Comparison with White's modified QG model*

The quasi-geostrophic model developed here (2.6) is identical to the classical three-dimensional one, except for the reduced-gravity boundary equation (the evolution equation of q_-). I am not aware of a similar development in the literature. However, the modified quasi-geostrophic model derived by White (1977), which incorporates non-Boussinesq effects, is *formally* similar to the present one *at the lower boundary*. It seems important to analyse to what extent the models differ. White (1977) considered two modifications to the Boussinesq QG model: (*a*) The reference density variation with height is included, $\rho_{0,z} = -\left(g^{-1}N^2 + gc_s^{-2}\right)\rho_0$, which gives an extra term in the definition of q. (b) The hydrostatic balance is written as $\partial_z p - g^{-1}N^2 p = -N^2\zeta$, instead of (2.2$a$); using *rigid* and *horizontal* top and bottom, the boundary condition $\mathrm{D}\zeta/\mathrm{D}t = 0$ reads $\mathrm{D}\left(\psi_{,z} - g^{-1}N^2\psi\right)/\mathrm{D}t = 0$ at $z = 0$ and $z = -H_r$. The present model, on the other hand, uses $\mathrm{D}\psi_{,z}/\mathrm{D}t = 0$ at $z = 0$ (rigid top) and $\mathrm{D}\left(\psi_{,z} - g_b^{-1}N^2\psi\right)/\mathrm{D}t = 0$ at $z = -H_r$ (soft bottom). Even though at the lower boundary both models satisfy a similar equation, the physical interpretation is quite different, because $g^{-1} \ll g_b^{-1}$. For the upper ocean, White's modifications are negligible: s_W ($:= g^{-1}N^2H_r$) and s_C ($:= gc_s^{-2}H_r$) are of the order of 10^{-3} or less. The extra term in the energy equation (2.7) is the external potential energy due to the deformation of the interface, whereas in White's model the Boussinesq-approximation internal potential energy is modified, because ζ^2 is proportional to $\left(\psi_{,z} - g_b^{-1}N^2\psi\right)^2$, and there is an extra contribution $c_s^{-2}p^2$, due to the compressibility (Blumen 1978; Kushner & Shepherd 1995).

If the upper boundary were also free in the present model, with g_0 the buoyancy variation across that boundary, then the top condition would be $\mathrm{D}\left(\psi_{,z} + g_0^{-1}N^2\psi\right)/\mathrm{D}t = 0$ at $z = 0$. Notice the sign difference with White's model, which is quite important: solving for Eady's problem with his modified boundary conditions but keeping $\rho_0 = \text{const.}$, White finds a singularity of the eigenvalue c at $\kappa = s_W$, a singularity which is absent in the present model with one or two soft boundaries. Blumen (1978) included the variation of $\rho_0(z)$, which requires an exponential $U(z)$, finding that the singularity disappears but that, instead, there is always a long-wave cutoff of the instability,† whereas in the present model there are growing perturbations with arbitrary small κ (at $v = 0$). Even for weak stratifications and short perturbations in both models, making $s_W = s \to 0$ and $\kappa^2 \gg s$, the dispersion relations that result for White's model and the present one are also found to differ.

5. Energetics

Consider any x-independent basic flow and an $O(a)$ perturbation, where it is formally assumed that $a \to 0$. The total streamfunction can be expanded as

$$\begin{array}{lllllll} \psi & = \Psi(y,z) + & \psi'(x,y,z,t) + & \Psi'(y,z,t) + & \psi''(x,y,z,t) + & \cdots \\ O: & 1 & a & a^2 & a^2 & a^3 \end{array}$$

(McPhaden & Ripa 1990; Ripa 1992b). The first term represents the prescribed basic flow (which is a trivial nonlinear solution of the equations of motion, because it is x- and t-independent). For the second term, a normal mode perturbation is selected $\psi' = \mathrm{Re}\left(a\,\hat{\psi}(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)}\right)$. The third term represents the $O(a^2)$ correction to the mean flow produced by the perturbation rectification, which is calculated from the x-average of the model equations (2.6), namely

$$Q'_{n,t} + \overline{\left[\psi'_n, q'_n\right]} = 0, \tag{5.1}$$

where $Q' = \left(\partial_y^2 + f_0^2N^{-2}\partial_z^2\right)\Psi'$, etc.; constancy of Kelvin circulations further requires $\Psi'_{,yt} = 0$ at $y = 0$, W and $-H_r < z < 0$. Finally, the fourth term is the harmonic correction of the perturbation, which will not be considered here.

The total free energy is $\mathscr{E} \sim \sum_{n=0}^{\infty}\mathscr{E}^{(n)}a^n$, and clearly $\mathrm{d}\mathscr{E}/\mathrm{d}t = 0$ implies $\mathrm{d}\mathscr{E}^{(n)}/\mathrm{d}t = 0$ $\forall n$. The first law, $\mathrm{d}\mathscr{E}^{(0)}/\mathrm{d}t = 0$, is trivial because $\mathscr{E}^{(0)}$ is only a functional of the time-independent basic state Ψ, whereas $\mathscr{E}^{(1)} = 0$ is easily satisfied by virtue of the $\mathrm{e}^{\mathrm{i}kx}$

† Blumen chose $s_W + s_C = 1$, but it can be shown that the long-wave cutoff is present for arbitrary values of these two parameters.

factor. (For a more general horizontal domain D, though, a growing or decaying eigen-perturbation must be such $\mathscr{E}^{(1)} = 0$, which is actually a restriction on the shape of the linear eigenfunction.) At second order in a, $\mathscr{E}^{(2)} = \mathscr{E}_W^{(2)} + \mathscr{E}_M^{(2)}$, where $\mathscr{E}_W^{(2)} \equiv \mathscr{E}[\psi'] = \frac{1}{2}\langle \lfloor \nabla\psi' \cdot \nabla\psi' \rceil^z + \cdots \rangle$ and $\mathscr{E}_M^{(2)} \equiv \delta\mathscr{E}[\Psi, \Psi'] = \langle \lfloor \nabla\Psi \cdot \nabla\Psi' \rceil^z + \cdots \rangle$. Traditionally, quasi-geostrophic instability is studied in terms of energy transfers between the perturbation and the mean flow, so that

$$\frac{\mathrm{d}\mathscr{E}_W^{(2)}}{\mathrm{d}t} + \frac{\mathrm{d}\mathscr{E}_M^{(2)}}{\mathrm{d}t} = 0.$$

In order to evaluate each term, quite a bit of algebra can be saved by using expression (2.8) for the first variation of the free energy $\delta\mathscr{E}[\psi, \delta\psi]$.

First, the rate of change of the perturbation energy $\mathrm{d}\mathscr{E}[\psi']/\mathrm{d}t = \delta\mathscr{E}[\psi', \psi'_{,t}]$ yields $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t = -\left\langle \lfloor \psi' q'_{,t} \rceil^z + \psi'_+ q'_{+,t} + \psi'_- q'_{-,t} \right\rangle$ plus terms proportional to $\psi'|_{X\in\partial D_i}\,\gamma'_{i,t}$ which vanish because of the constancy of Kelvin circulations. Now, each term can be transformed as in $-\left\langle \psi' q'_{,t} \right\rangle = \left\langle \psi' U q'_{,x} \right\rangle = -U \left\langle v' q' \right\rangle$, using first the equation for $q'_{,t}$ and then making a partial integration, which yields

$$\frac{\mathrm{d}\mathscr{E}_W^{(2)}}{\mathrm{d}t} = -\left\langle \lfloor (U-\alpha)\, v' q' \rceil^z + (U_+ - \alpha)\, v'_+ q'_+ + (U_- - \alpha)\, v'_- q'_- \right\rangle, \tag{5.2}$$

where α is any constant. Consequently, the term proportional to α vanishes identically: this constitutes the generalization, to the problem with a free boundary, of a theorem originally derived by Bretherton (1966) for the system with rigid and horizontal boundaries. It is clear that to classify instabilities by the relative size of the different terms in (5.2) can be misleading: one might choose α as the value of U in a certain region, and then apparently that region does not contribute to the energetics. Only the total value $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t$ has a clear physical meaning.

Secondly, evaluation of $\delta\mathscr{E}[\Psi, \Psi']$ yields

$$\mathscr{E}_M^{(2)} = -\left\langle \lfloor \Psi Q' \rceil^z + \Psi_+ Q'_+ + \Psi_- Q'_- \right\rangle,$$

modulo the (constant) circulation changes $\Psi\, \Gamma'_i$ at $y = 0, W$. Taking the time derivative and using (5.1) it follows that $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t + \mathrm{d}\mathscr{E}_M^{(2)}/\mathrm{d}t = 0$, as desired. (This form of deriving the energy equations using but the form of the first variation (2.8) can also be used for non-parallel flow instability, in a more general domain D.)

From the linearized equations $Q_{n,y}\overline{v'_n q'_n} = -\frac{1}{2}\partial_t \overline{q'^2_n}$, and therefore, if $Q_{n,y} \neq 0$ the right-hand side of (5.2) equals $\mathrm{d}\mathscr{C}_W^{(2)}/\mathrm{d}t$ where the wave Casimir $\mathscr{C}_W^{(2)} = \delta^2\mathscr{C}[\Psi, \psi']$ is the $O\left(a^2\right)$ contribution to (3.2*b*). Consequently, the $O\left(a^2\right)$ energy conservation law, $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t + \mathrm{d}\mathscr{E}_M^{(2)}/\mathrm{d}t = 0$, is equivalent to the lowest-order pseudoenergy conservation law $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t + \mathrm{d}\mathscr{C}_W^{(2)}/\mathrm{d}t = 0$. However, in order to evaluate the variation of the mean flow energy $\mathscr{E}_M^{(2)}$ it is necessary to know the $O\left(a^2\right)$ fields, whereas the wave Casimir $\mathscr{C}_M^{(2)}$ is a functional of only the $O(a)$ perturbation (McIntyre & Shepherd 1987; Ripa 1992*b*).

For the case (3.4), the wave rectification produces a mean-flow variation $Q' \equiv 0$ (because $\beta = 0$) and

$$Q'_\pm(y,t) = \pm\tfrac{1}{4}\left|a\hat{q}_+\right|^2 l L^2 U_s^{-1} \sin 2ly \left(\mathrm{e}^{2k\,\mathrm{Im}\,ct} - 1\right);$$

the corresponding streamfunction $\Psi(y,z,t)$ is obtained from equations (A 4) and (A 6). In order to use equation (5.2) for the rate of change of the perturbation energy, note that $q' = 0$ and $\overline{v'_\pm q'_\pm} = \mp\frac{1}{4}\left|a\hat{q}_+\right|^2 U_s^{-1} L^2 (2k\,\mathrm{Im}\,c) \sin^2 ly\, \mathrm{e}^{2k\,\mathrm{Im}\,ct}$. From (5.2) it

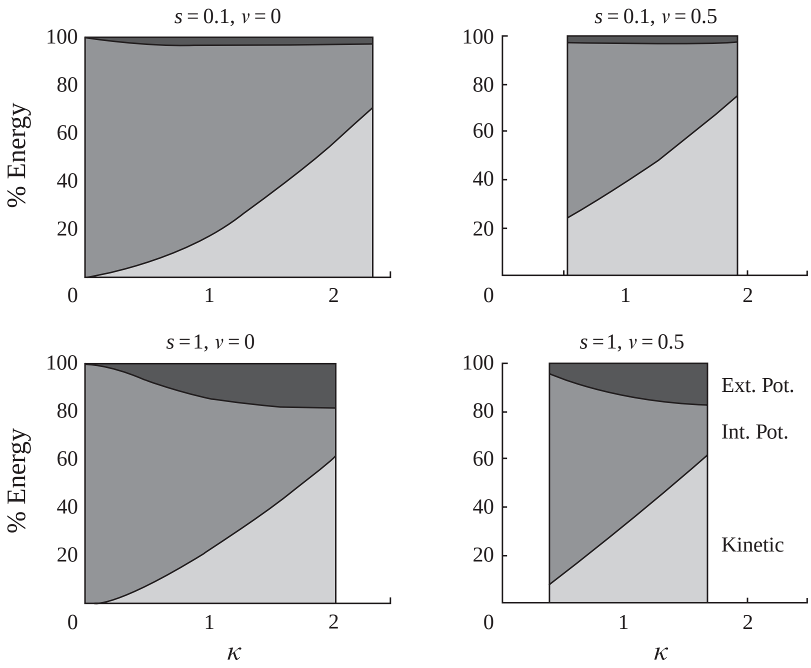

FIGURE 8. Energy partition of growing perturbations into kinetic $\iiint (\nabla\psi')^2$, internal potential $\iiint f_0^2 N^{-2} (\partial_z\psi')^2$, and external potential $\iint f_0^2 g_b^{-1} (\psi')^2 |_{z=-H_r}$.

follows that $\mathrm{d}\mathscr{E}_W^{(2)}/\mathrm{d}t = \frac{1}{8}L^2\,|a\hat{q}_+|^2\,(2k\,\mathrm{Im}\,c)\,\mathrm{e}^{2(k\,\mathrm{Im}\,c)t}$; by simple integration in time it is then found that

$$\mathscr{E}_W^{(2)} = \tfrac{1}{2}\langle\lfloor(\nabla\psi')^2 + f_0^2N_r^{-2}(\partial_z\psi')^2\rceil^z + f_0^2g_b^{-1}\psi_-'^2\rangle$$
$$= \tfrac{1}{8}L^2\,|a\hat{q}_+|^2\,\mathrm{e}^{2(k\,\mathrm{Im}\,c)t}.$$

A more tedious calculation gives the partition of this energy into its three parts (kinetic, internal potential, and external potential) as $[A-B, A+B, 1-2A]\times\mathscr{E}_W^{(2)}$, where $A = \tau(s+\kappa^2)/[2\kappa(\kappa\tau+s)]$ and $B = [s+(2-\nu-s)\tau\kappa-\kappa^2]/[2(\kappa\tau+s)]$. Notice that the fraction of external potential energy, $1-2A$, is independent of ν.

Figure 8 shows how is the energy partitioned in a growing perturbation. At low wavenumbers, near the long-perturbation cutoff, the energy is mostly internal potential (and not kinetic, as classical descriptions of baroclinic instability account), whereas at high wavenumbers, near the short-perturbation cutoff, kinetic energy is also important. As $s\to 0$, the maximum fraction of external potential energy equals $s/3$ and it is reached at $\kappa = \sqrt[4]{15s}$. On the other hand, at strong stratifications, $s \gg 1$, most of the perturbation energy is concentrated in this form, i.e. on the deformation of the interface. This is consistent with the results of § 4, where it was shown that the effect of the perturbation on the interface is more important for finite s.

6. Conclusions

The upper part of the ocean displays a myriad of processes, not all of them fully understood yet. A vertically well mixed (by, say, the wind action) but horizontally inhomogeneous layer restratifies through the radiation of Poincaré waves (Tandon & Garrett 1995). The mean current that results from this process might, in turn, be baroclinically unstable, leading to a nonlinear process that could favour the subsequent mixing of the layer (Haine & Marshall 1998). This scenario is one of the motivations that leads towards the study of nonlinear baroclinic instability in a reduced-gravity model (i.e. with a soft lower boundary). The f-plane normal mode solution found by Fukamachi *et al.* (1995) and further generalized by Beron-Vera & Ripa (1997) has been thoroughly analysed here, by considering all possible values of the interior stratification s (relative to the buoyancy jump at the interface) and interface slope v (relative to the isopycnals basic slope; these parameters are defined in equation (3.5)). For wavenumbers close to the short-wave cutoff $\kappa_S(v, s)$, an important fraction of the mean flow energy is converted into 'external' potential energy of the perturbation, unless the internal stratification s is very small. For wavenumbers near the long-wave cutoff $\kappa_L(v, s)$, on the other hand, most of the perturbation energy is internal potential. It would be interesting to investigate the importance of this wavenumber dependence of the structure of growing perturbations on processes like the gas exchange with the deep ocean, discussed by Haine & Marshall (1998).

In order to apply this model to a more realistic situation, other processes need to be incorporated. For instance, the second layer may be allowed to be active, and then (2.6) for $q_n = q_-$ becomes the matching condition between both layers. Buoyancy flux at the surface and entrainment of lower layer water can be modelled as forcing terms in the right-hand side of model equations (2.6). The curl of wind stress can enter as Ekman pumping in (2.6) for $q_n = q_+$ or as a body force in the vorticity equation (2.6) for $q_n = q$. The interplay of these forcing mechanisms and the baroclinic instability process discussed here constitutes an important problem which goes beyond the goals of the present paper.

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Appendix. Evaluation of the streamfunction as a functional of the prognostic fields

Let $\delta q\,(\boldsymbol{x}, z, t)$, $\delta q_\pm\,(\boldsymbol{x}, t)$, and $\delta\gamma_i\,(z)$ be arbitrary perturbation fields in the channel $0 < y < W$. In order to calculate the corresponding streamfunction perturbation $\delta\psi\,(\boldsymbol{x}, z, t)$ the primary fields are expanded as

$$\left.\begin{aligned} \delta q\,(\boldsymbol{x}, z, t) &= \sum_{k,l,m} q^{k,l,m}\,(t)\,\mathrm{e}^{\mathrm{i}kx}\sin ly\,F_m\,(z)\,, \\ \delta q_\pm\,(\boldsymbol{x}, t) &= \sum_{k,l} q_\pm^{k,l}\,(t)\,\mathrm{e}^{\mathrm{i}kx}\sin ly, \\ \delta\gamma_i\,(z) &= \sum_m \gamma_i^m F_m\,(z)\,, \end{aligned}\right\} \qquad \text{(A 1)}$$

where $l = n\pi/W$ and

$$F_m(z) = \sqrt{\frac{2\mu_m^2 + 2s^2}{\mu_m^2 + s^2 + s}} \cos\frac{\mu_m z}{H_r}, \tag{A 2}$$

with $\mu_m \tan\mu_m = s$ ($m\pi < \mu_m < \left(m+\frac{1}{2}\right)\pi$), are the vertical structure functions of the Rossby waves for $\beta \neq 0$ and $U_b = 0$ (Ripa 1995). (The normalization constant gives the orthogonality conditions $\lfloor F_m F_r \rceil^z = \delta_{mr}$ and $H_r^2 \lfloor F_m' F_r' \rceil^z + s(F_m F_r)_- = \mu^2 \delta_{mr}$.)

First, the contribution of δq to the streamfunction is written as

$$\delta\psi(\boldsymbol{x}, z, t) = -L^2 \sum_{k,l,m} \left(\kappa^2 + \mu^2\right)^{-1} q^{k,l,m}(t)\, \mathrm{e}^{\mathrm{i}kx} \sin ly\, F_m(z), \tag{A 3}$$

where $\kappa = \sqrt{k^2 + l^2}L$. Similarly, the contribution of $\delta q_\pm$ to the streamfunction takes the form

$$\left.\begin{aligned} \delta\psi(\boldsymbol{x}, z, t) &= L^2 \sum_{k,l} [q_+^{k,l}(t)\, G_+^\kappa(z) + q_-^{k,l}(t)\, G_-^\kappa(z)] \mathrm{e}^{\mathrm{i}kx} \sin ly \\ G_+(z) &= -\frac{(\kappa + \tau s)\cosh\left(\kappa z/H_r\right)}{(\tau\kappa + s)\,\kappa} - \frac{\sinh\left(\kappa z/H_r\right)}{\kappa}, \\ G_-(z) &= -\frac{\cosh\left(\kappa z/H_r\right)}{(\tau\kappa + s)\cosh\kappa}, \end{aligned}\right\} \tag{A 4}$$

where $\tau = \tanh\kappa$. The $G_\pm$ and the F_m are not orthogonal; the contribution of the three diagnostic fields can be grouped in the form $\delta\psi(\boldsymbol{x}, z, t) = \sum_{k,l,m} \psi_{k,l,m}(t)\, \mathrm{e}^{\mathrm{i}kx} \sin ly\, F_m(z)$, with

$$\psi_{k,l,m}(t) = -L^2 \left(\kappa^2 + \mu^2\right)^{-1} [q^{k,l,m}(t) + F_+^m q_+^{k,l}(t) + F_-^m q_-^{k,l}(t)], \tag{A 5}$$

where $F_+^m = F_m(0)$ and $F_-^m = F_m(-H_r)$. Now, a term $\psi_{k,l,m}(t)\, \mathrm{e}^{\mathrm{i}kx} \sin ly\, F_m(z)$ does not satisfy the constraint of constant Kelvin circulation for $k = 0$. Therefore, to the $k = 0$ part of the above expansion must be added

$$\delta\psi(\boldsymbol{x}, z, t) = \cdots + \sum_{l,m} \psi_{0,l,m}(t)\, \frac{lL}{\mu} \frac{\mathrm{e}^{-\mu y/L} \mp \mathrm{e}^{-\mu(W-y)/L}}{1 \pm \mathrm{e}^{-\mu W/L}} F_m(z), \tag{A 6}$$

where $\cos lW = \pm 1$. Finally, the contribution of $\delta\gamma_i$ to the streamfunction is of the form

$$\delta\psi(y, z) = \sum_{m,\pm} \frac{L}{2\mu} \left(\gamma_1^m \pm \gamma_2^m\right) \frac{\mathrm{e}^{-\mu y/L} \mp \mathrm{e}^{-\mu(W-y)/L}}{1 \pm \mathrm{e}^{-\mu W/L}} F_m(z). \tag{A 7}$$

Notice that the corrections terms in (A 6) as well as (A 7) correspond to pressure and velocity fields identical to those of a superposition of coastally trapped Kelvin waves, with $k = 0$.

An alternative description of the $k = 0$ cases is use $\delta q = \sum \ldots \cos ly$, for which the $\delta\psi = \sum \ldots \cos ly$ has a vanishing circulation. However, this expansion converges less rapidly than the one presented here.

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